B. Math. (Hons.) 2nd year Midsemestral Exam
Rings and Modules
21st February 2023
Instructor — B. Sury
Each Question carries 10 points

Q 1. Let F be a field. Prove that the groups $(F, +)$ and $(F \setminus \{0\}, \times)$ are not isomorphic.

OR

Q 1. Let R be a (possibly noncommutative) ring with unity with the property that whenever two elements a, b satisfy $ab = 1$, we also have $ba = 1$. Let $f, g \in R[X]$, such that $fg = 1$ in $R[X]$. Prove that $gf = 1$ as well.

Q 2.
(a) Let R be a (commutative) local ring with unity. Show that the only idempotents in R are 0 and 1.
(b) Let R be a commutative ring with unity. If I, J are ideals satisfying $I + J = R$, prove that $I^m + J^n = R$ for all positive integers m, n.

OR

Q 2. If R is a commutative ring with unity, and $f = \sum_{i=0}^{n} a_i X^i$ is a unit, prove that $a_1, \cdots, a_n$ are nilpotent.

Q 3. Let G be a non-trivial finite group. Let R be any commutative ring with unity. Prove that the group ring $R[G]$ has nontrivial zero divisors.

OR

Q 3. If $f : A \to B$ is a ring homomorphism, give examples of ideals J_1, J_2 in B to show that the containment $(J_1 : J_2)^c \subset J_1^c : J_2^c$ can be strict.

Q 4. Let R be a commutative ring with unity in which there are only finitely many nilpotent elements. Suppose $f(X) = \sum_{i=0}^{\infty} a_i X^i \in R[[X]]$ where each a_i is nilpotent. Prove that f is nilpotent.

OR

Q 4. If R is a commutative ring with unity, and I, J are ideals such that $I + J \neq R$, prove that the natural ring homomorphism $R \to R/I \times R/J$; $x \mapsto (x + I, x + J)$ is not surjective.

Q 5. Let R be a commutative ring with unity. For ideals I, J in R, prove that the sets $V(I) := \{P \text{ prime ideal: } P \supseteq I\}$ and $V(J) := \{P \text{ prime ideal: } P \supseteq J\}$ coincide if, and only if, $\sqrt{I} = \sqrt{J}$.

OR

Q 5. Let R be a (commutative) integral domain with quotient field F. For each prime ideal P, the localization R_P can be considered a subring of F containing R. Prove that the intersection of all $R_{\mathfrak{m}}$ as $\mathfrak{m}$ runs over the maximal ideals of R, equals R.

Q 6. For any positive integers k, n, prove that $X_1^n + X_2^n + \cdots + X_k^n + 3$ is an irreducible element of $\mathbb{Z}[X_1, \cdots, X_k]$. State precisely any of the results that you use without proof.

OR

Q 6. Prove that the polynomial $X^{20020} - 10010X^{2002} + 110$ does not take the values ± 33 for any integer values of X.

Q 7. Does the equation $2.7 = \sqrt{14}.\sqrt{14}$ in the ring $R = \mathbb{Z}[\sqrt{14}]$ imply that R is not a UFD? Explain why or why not.

OR

Q 7. Determine all the integer solutions of the equation $x^2 + 2 = y^3$. You may assume that $\mathbb{Z}[\sqrt{-2}]$ is a UFD.